DEPT. OF MATH. UNIVERSITY OF OSLO
PURE MATHEMATICS
ISBN 82–553–1343–5 No. 11
ISSN 0806–2439 May 2002

ON DERIVATIVES OF CLAIMS IN COMMODITY AND ENERGY MARKETS USING A MALLIAVIN APPROACH

FRED E. BENTH, LARS O. DAHL, AND KENNETH H. KARLSEN

ABSTRACT. In this paper we investigate the recently introduced Malliavin approach compared to more classical approaches to find sensitivities of options in commodity and energy markets. The Malliavin approach has been developed in the paper [9] and [10]. In commodity and energy markets, some special dynamics for the underlying security and some new products different from Black & Scholes markets are encountered. In addition to investigating the numerical values of the expressions by conventional Monte Carlo (MC) and quasi-Monte Carlo (QMC) methods, we apply an adaptive approach developed in the papers [7] and [8]. This adaptive method is also applied to the so called Localized Malliavin approach developed in the paper [10]. The numerical results show that we can get substantial variance reduction by choosing sophisticated methods for the simulations, and that the Malliavin approach is a very powerful tool for formulating the sensitivity estimators.

1. INTRODUCTION

In commodity and energy markets, the underlying product of a derivative contract may be either the spot or the forward/futures contract on the spot. A much used model for spot prices in commodity and energy markets is Schwartz' mean-reverting model (see [21] and [15]). Formulated in a risk-neutral world it has the dynamics,

$$dS(t) = \alpha(\mu - \lambda - \ln S(t))S(t)\,dt + \sigma S(t)\,dW(t)\ . \tag{1}$$

Here, α is the mean-reversion rate, σ the volatility, e^{μ} the long-term level for the spot price and λ the market price of risk. $W(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_t$ is the augmentation with respect to P of the filtration generated by W, $0 \leq t \leq T < \infty$. If we introduce $\gamma = \alpha(\mu - \lambda) - \sigma^2/2$, we may write $S(t) = \exp(X(t))$ for the Ornstein-Uhlenbeck process

$$dX(t) = (\gamma - \alpha X(t))\,dt + \sigma dW(t)\ , \tag{2}$$

Date: 1st May 2002.
Key words and phrases. Options, options in commodity and energy markets, sensitivity measures, hedging, Malliavin derivative, Monte Carlo simulation.

with $X(0) = \ln x, S(0) = x$. The process $X(t)$ has an analytical expression

$$X(t) = \mathrm{e}^{-\alpha t} \ln x + \gamma(1 - \mathrm{e}^{-\alpha t}) + \sigma \int_0^t \mathrm{e}^{-\alpha(t-s)} \, dW(s) \, .$$

This analytical expression is useful when calculating derivatives of claims.

Prices of forward instruments can be derived in an arbitrage-free way from the spot price (see e.g. [5, 2, 19]). However, motivated from the Heath-Jarrow-Morton approach in interest rate theory, one may instead write down the risk-neutral dynamics of the forward price directly. We assume the dynamics of the forward contract on the spot is given in the risk-neutral world as

$$dF(t,T) = \sigma(t,T) F(t,T) \, dW(t), \quad F(0,T) = x(T) \, , \tag{3}$$

where $x(T)$ is todays forward curve. We assume σ is an integrable function such that $\int_0^T \sigma^2(t,T) \, dt < \infty$, which means that $t \to F(t,T)$ is a martingale. An explicit representation of $F(t,T)$ is

$$F(t,T) = x(T) \exp\Big(-\frac{1}{2} \int_0^t \sigma^2(s,T) \, ds + \int_0^t \sigma(s,T) \, dW(s)\Big) \, . \tag{4}$$

A frequently used volatility structure in commodity and energy markets is given by

$$\sigma(t,T) = \sigma \mathrm{e}^{-\alpha(T-t)} \, . \tag{5}$$

This specification is motivated from the mean-reverting model of Schwartz for spot prices, which implies this volatility structure for the forward price (see e.g. [5, 2]). When considering claims on spot prices, the dynamics in (1) will be assumed. When we on the other hand analyze claims on the forward, we use the forward dynamics (3) with volatility structure as in (5).

We focus in this paper on derivatives of the price of different claims with respect to different parameters in the underlying. In sec. 2 we consider derivatives of European options on spot prices of the commodity and energy market. In particular we find expressions for delta, gamma and vega. In sec. 3 we find the derivatives of European options on forward prices. The same parameters as for the spot are calculated. We advance in sec. 4 by looking at path dependent options, in particular the European-style arithmetic average Asian option, which produces a multidimensional problem. The sensitivities we find by differentiating the prices of such claims are extensively used in the process of hedging contracts of this type. Practicians need to have a well developed intuition of the dependence of their position on the movements and events in the market, and a range of literature give interpretations of the parameters. See e.g. [14], [1], [22].

The derivatives can be expressed in various ways, depending on how they are deduced and the assumptions made in the deduction. We will mainly focus on the Malliavin approach recently introduced in the papers [9] and [10]. The approach uses the Malliavin derivative together with properties of the Skorohod integral to produce formulas for the derivatives of options. These formulas are expressed in terms of the expectation of the option's payoff multiplied with some random variable which is (usually) a function of the underlying. A neat feature of the Malliavin approach is that this random variable is not dependent on the actual option (that is, f), but on the underlying product. This means that Monte Carlo based algorithms for numerically evaluating derivatives can be made for general options, and not specifically for each option. This is in contrast to the direct method (also called infinitesimal perturbation analysis (IPA)), where one must deduce individual expressions for each payoff function and underlying contract since the derivative is expressed in terms of the differential of the payoff function. See e.g. [12] for an overview of the direct method or [4] for deduction of sensitivity expressions in the geometric Brownian motion (GBM) setting. Another conventional method frequently used is the so-called density approach, which relies on the existence of a density of the underlying product. This method expresses the derivative in terms of the option's payoff multiplied with a random variable, very much similar to the Malliavin approach. However, the density method deduces *one* such random variable, while the Malliavin method provides a flexible class of variables. Also, when for instance the option is of Asian-type, there exists no density, and the density approach fails. The Malliavin approach handles this type of products, demonstrating the flexibility of the method.

To compare the Malliavin approach and the alternative methods, we deduce sensitivities by the direct method and the density approach, whenever these methods can be used. In this way we are able to illustrate both the flexibility gained with the Malliavin approach in that it can be applied with success where other methods fail, and to investigate how different numerical methods apply to the different derivative approaches. A further improvement of the Malliavin approach is the localized version introduced in [9, 10]. The Localized Malliavin approach uses the Malliavin methods around point where the payoff function is not smooth, and direct method outside.

The contribution of the paper is twofold. First, we present formulas for derivatives of options in commodity and energy markets based on the Malliavin approach and compare these with the corresponding expression found by conventional methods. A large portion of the present paper consists of such formulas. Secondly, we investigate effective numerical methods for estimating the sensitivities based on the different formulas derived in the first part. We have implemented a quasi-Monte Carlo method based on the Halton[1] low discrepancy sequence as a basis for this exploration. Furthermore we have adjusted an adaptive method developed

[1]The Halton sequence was first presented in [13]. In this paper we are using an extension of the Halton sequence denoted the Halton leaped sequence. It was presented in [16], together with good leap values. We have used the leap value 31 in the numerical experiments.

by the authors in [7] and [8] to the current problem, resulting in an adaptive QMC method. The numerical tests are performed both with and without the adaptive method in order to investigate the effect of applying this. Furthermore we investigate the difference in numerical stability and convergence speed for the estimators deduced by the three approaches; Malliavin, Localized Malliavin and Forward Difference. We know from previous work that the adaptive method is able to perform very well for low dimensional integrals, and the results of this paper show that also in the current setting it gives enormous speedup for many of the problems. The numerical results furthermore verify that the Malliavin approach is the best alternative for finding sensitivities when the payoff function of the option is discontinuous. Our numerical results, somewhat surprisingly, also show that the Local Malliavin approach does not give an estimator with lower variance than the Malliavin approach, but almost identical. However, we have not numerically tested the Localized Malliavin approach to calculate the gamma, and it is likely that the Localized Malliavin approach is able to perform better in this setting.

We emphasize that the adaptive method in this context is not "competing" with the Malliavin approach, but a supplement used to refine the use of Monte Carlo sampling points also in the Malliavin and Local Malliavin context. The results of the simulations are collected in sec. 7.

2. Derivatives of European options on commodity and energy spots

Consider a European option with maturity T and payoff $f(S(T))$, where $f \in L^2(\mathbb{R})$ and $\mathbb{E}\big[f(S(T))^2\big] < \infty$. The price of the option is

$$u = \mathrm{e}^{-rT}\mathbb{E}\big[f(S(T))\big] \,. \tag{6}$$

Recall that the spot is formulated directly in the risk-neutral setting. We shall use the notation $u(x)$ when we consider the price as a function of the strike spot price x, and $u(\sigma)$ when we consider the price as a function of the volatility σ. For simplicity, we will assume throughout the rest of the paper that the risk-free interest rate is zero, i.e. $r = 0$.

We are interested in calculating the delta, $u'(x)$, of the option price. Before we go to the Malliavin deduction, let us present the direct approach, also called infinitesimal perturbation analysis (IPA). Provided we can move differentiation into the expectation (sufficient conditions for this are given in e.g. [17], [12], [4]), we can simply write the delta as

$$\begin{aligned} u'(x) &= \mathbb{E}\big[f'(S(T))\frac{dS(T)}{dx}\big] \\ &= \mathbb{E}\big[f'(S(T))S(T)\frac{\mathrm{e}^{-\alpha T}}{x}\big] \,. \end{aligned}$$

When $f(x) = (x-K)^+$, we have $f'(x) = \mathbf{1}_{x>K}$. This method however gets into trouble if f is discontinuous, for example $f(x) = \mathbf{1}_{x>K}$. By the same argument, the direct method is not suited if we want to find the second derivative (e.g. gamma) of this f. Furthermore, the algorithmic treatment of this approach is depending on the specific payoff function f, resulting in individual implementations for each payoff function and each instrument. The method is therefore not very flexible in this context. As we shall see, The Malliavin approach circumvent all these limitations. An other approach that has the ability to circumvent this, but gives a bias, is the finite difference (FD) method. The derivative is then simply found by the estimator

$$\frac{u(x+h)-u(x)}{h} \,,$$

where we use the same Brownian trajectories for both function evaluations to reduce the variance of the estimator. The parameter h should be small to reduce the bias of the estimator, but a smaller h results in an estimator with larger variance. We use h in the range $[0.1, 0.001]$ percent of x. See [4] for a discussion on finding the optimal h for the FD estimator.

Next we turn to the Malliavin approach. However, before we can state the propositions on the sensitivities , we need to introduce the Malliavin derivative and state some useful properties of the Skorohod integral.

2.1. **Some results from the Malliavin Calculus.** Let $\mathcal{C}$ be the set of cylinder functions on the probability space, e.g. the set of random variables of the form

$$G = g\big(\int_0^\infty h_{1(t)}\, dW(t), \ldots, \int_0^\infty h_{n(t)}\, dW(t)\big) \,,$$

where $g \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$, and $h_i \in L^2(\Omega \times \mathbb{R})$. The Malliavin derivative of $G \in \mathcal{C}$ is the process $D_t X$ defined as

$$D_t G = \sum_{i=1}^n \frac{\partial g}{\partial x_i} h_i(t) \,.$$

Introducing the Banach space $\mathbf{D}^{1,2}$ as the completion of $\mathcal{C}$ with respect to the norm

$$\|G\|_{1,2}^2 = \mathbb{E}\big[G^2\big] + \mathbb{E}\big[\int_0^T (D_t G)^2\, dt\big] \,,$$

we can extend D to be a closed linear operator defined in $\mathbf{D}^{1,2}$. If Y is an Ito-integrable process, then the Malliavin derivative of $\int_0^T Y(s)\,dW(s)$ is

$$D_t \int_0^T Y(s)\,dW(s) = Y(t)\mathbf{1}_{t<T}\ .$$

Furthermore, if $Y \in \mathbf{D}^{1,2}$ and g is a continuously differentiable function with bounded derivative, then $g(Y) \in \mathbf{D}^{1,2}$, and the chain rule holds for the Malliavin derivative:

$$D_t g(Y) = g'(Y) D_t Y\ .$$

We proceed with some results on the Skorohod integral, a stochastic integral for a class of anticipating stochastic processes $Y(t)$ which we denote $\int Y(s)\,\delta W(s)$. It is defined as the adjoint operator of D in the following manner: Let Y be a stochastic process. Then Y is said to be Skorohod integrable if for any $G \in \mathbf{D}^{1,2}$ we have

$$\mathbb{E}\big[\int_0^T Y(t) D_t G\,dt\big] \leq C\|G\|_{1,2}\ ,$$

where C is a constant depending on Y. The Skorohod integral of Y, $\int Y(s)\,\delta W(s)$, is defined by the following duality relation: For any $G \in \mathbf{D}^{1,2}$

$$\mathbb{E}\big[G \int_0^T Y(t)\,\delta W(t)\big] = \mathbb{E}\big[\int_0^T Y(t) D_t G\,dt\big]\ .$$

We state two basic properties of the Skorohod integral, which will be used frequently in what follows. The first proposition tells us that Skorohod integration is a true generalization of the Ito integral:

Proposition 2.1. *Let Y be an Ito integrable stochastic process. Then, Y is integrable in the sense of Skorohod and*

$$\int_0^T Y(t)\,\delta W(t) = \int_0^T Y(t)\,dW(t)\ .$$

The Skorohod integral possesses an integration-by-parts property:

Proposition 2.2 (Integration-by-parts)**.** *Let $G \in \mathbf{D}^{1,2}$ be an $\mathcal{F}_T$-adapted random variable. Then, for any Skorohod integrable stochastic process Y*

$$\int_0^T GY(t)\,\delta W(t) = G\int_0^T Y(t)\,\delta W(t) - \int_0^T Y(t) D_t G\,dt\ .$$

The proofs of the above propositions can be found in e.g. [18], where a complete account of the Malliavin Calculus can be found.

2.2. **Derivatives of options on spot.** We now turn our attention to the computation of expressions of option derivatives using the Malliavin approach. We remark that many of the results below follow from the general results in [9, 10]. For the sake of clarity we have chosen to derive the expressions for the specific models we have in mind. Introduce the set of functions

$$\Gamma_T = \{a \in L^2([0,T]) | \int_0^T a(t)\, dt = 1\} .$$

Then,

Proposition 2.3 (Delta by the Malliavin approach)**.** *The delta of $u(x)$ can be represented as*

$$u'(x) = \frac{1}{\sigma x}\mathbb{E}\big[f(S(T)) \int_0^T a(t) e^{-\alpha t}\, dW(t)\big] ,$$

where $a(t) \in \Gamma_T$.

Proof. Assume first that f is continuously differentiable with bounded derivative. It can then be shown that differentiation and expectation commutes, and thus

$$\begin{aligned} u'(x) &= \mathbb{E}\big[f'(S(T))\frac{\partial}{\partial x}S(T)\big] \\ &= x^{-1}\mathrm{e}^{-\alpha T}\mathbb{E}\big[f'(S(T))S(T)\big] . \end{aligned}$$

We have used that $\partial S(T)/\partial x = x^{-1}\mathrm{e}^{-\alpha T}S(T)$. The Malliavin derivative of the spot price is

$$D_t S(T) = \mathrm{e}^{X(T)} D_t X(T) = S(T)\sigma \mathrm{e}^{-\alpha(T-t)}\mathbf{1}_{\{t<T\}} .$$

Choose a function $a(t) \in \Gamma_T$. Integrating both sides above give

$$S(T) = \sigma^{-1}\mathrm{e}^{\alpha T}\int_0^T a(t)\mathrm{e}^{-\alpha t} D_t S(T)\, dt .$$

Using the properties of the Skorohod integral and the Malliavin derivative, this yields,

$$\begin{aligned} u'(x) &= \frac{1}{x\sigma}\mathbb{E}\big[\int_0^T f'(S(T))D_t S(T) a(t)\mathrm{e}^{-\alpha t}\, dt\big] \\ &= \frac{1}{x\sigma}\mathbb{E}\big[\int_0^T D_t f(S(T)) a(t)\mathrm{e}^{-\alpha t}\, dt\big] \\ &= \frac{1}{x\sigma}\mathbb{E}\big[f(S(T))\int_0^T a(t)\mathrm{e}^{-\alpha t}\, dW(t)\big] . \end{aligned}$$

By a density argument the formula can be extended to $f \in L^2$. See [9] for details on this. $\square$

In all the following deductions of option derivatives using the Malliavin approach we shall use the method above with first assuming smooth payoff functions, and then passing to the limit by a density argument. This will from now on be done without being explicitly stated.

If we choose $a(t) = \mathrm{e}^{2\alpha t}/\int_0^T \mathrm{e}^{2\alpha t}\,dt = 2\alpha\mathrm{e}^{2\alpha t}/(\mathrm{e}^{2\alpha T}-1)$, we get

$$u'(x) = \frac{2\alpha}{\sigma x(\mathrm{e}^{2\alpha T}-1)}\mathbb{E}\big[f(S(T))\int_0^T \mathrm{e}^{\alpha t}\,dW(t)\big]\ .$$

But

$$\begin{aligned}\int_0^T \mathrm{e}^{\alpha t}\,dW(t) &= \sigma^{-1}\mathrm{e}^{\alpha T}\cdot\sigma\int_0^T \mathrm{e}^{-\alpha(T-t)}\,dW(t)\\ &= \sigma^{-1}\mathrm{e}^{\alpha T}\big(X(T)-\mathrm{e}^{-\alpha T}\ln x-\gamma(1-\mathrm{e}^{-\alpha T})\big)\\ &= \sigma^{-1}\big(\mathrm{e}^{\alpha T}X(T)-\ln x-\gamma(\mathrm{e}^{\alpha T}-1)\big)\ ,\end{aligned}$$

where $\gamma = \alpha(\mu-\lambda)-\sigma^2/2$. Hence,

$$u'(x) = \mathbb{E}\big[f(S(T))\frac{2\alpha}{x\sigma^2(\mathrm{e}^{2\alpha T}-1)}\big(\mathrm{e}^{\alpha T}\ln S(T)-\ln x-\gamma(\mathrm{e}^{\alpha T}-1)\big)\big]\ . \tag{7}$$

If we differentiate the delta of $u(x)$, we find the gamma:

Proposition 2.4 (The Malliavin approach)**.** *The gamma of $u(x)$ can be represented as*

$$u''(x) = \frac{1}{\sigma^2x^2}\mathbb{E}\big[f(S(T))\big\{Z(T)^2-\sigma Z(T)-\int_0^T a^2(t)e^{-2\alpha t}\,dt\big\}\big]\ ,$$

where $Z(T) = \int_0^T a(t)e^{-\alpha t}\,dW(t)$ and $a(t)\in\Gamma_T$.

Proof. From Prop. 2.3 we have

$$u'(x) = \frac{1}{\sigma x}\mathbb{E}\big[f(S(T))Z(T)\big]$$

for

$$Z(T) = \int_0^T a(t)\mathrm{e}^{-\alpha t}\,dW(t)\ .$$

Hence,

$$\begin{aligned}u''(x) &= \frac{d}{dx}\frac{1}{\sigma x}\mathbb{E}\big[f(S(T))Z(T)\big]\\ &= \frac{-1}{\sigma x^2}\mathbb{E}\big[f(S(T))Z(T)\big]+\frac{1}{\sigma x}\mathbb{E}\big[f'(S(T))\frac{\partial}{\partial x}S(T)\cdot Z(T)\big]\ .\end{aligned}$$

We investigate the second expectation: Using that $\frac{\partial}{\partial x}S(T) = x^{-1}\mathrm{e}^{-\alpha T}S(T)$, and

$$S(T) = \sigma^{-1}\mathrm{e}^{\alpha T}\int_0^T a(t)\mathrm{e}^{-\alpha t}D_t S(T)\,dt\ ,$$

we obtain

$$\begin{aligned}\mathbb{E}\big[f'(S(T))\frac{\partial}{\partial x}S(T)\cdot Z(T)\big] &= \frac{1}{\sigma x}\mathbb{E}\big[\int_0^T f'(S(T))D_t S(T)a(t)\mathrm{e}^{-\alpha t}Z(T)\,dt\big]\\ &= \frac{1}{\sigma x}\mathbb{E}\big[\int_0^T D_t f(S(T))a(t)\mathrm{e}^{-\alpha t}Z(T)\,dt\big]\\ &= \frac{1}{\sigma x}\mathbb{E}\big[f(S(T))\int_0^T a(t)\mathrm{e}^{-\alpha t}Z(T)\delta W(t)\big]\ ,\end{aligned}$$

where δW mean the Skorohod integral, which is present since $Z(T)$ is anticipating. By the integration-by-parts formula for Skorohod integrals,

$$\begin{aligned}\int_0^T a(t)\mathrm{e}^{-\alpha t}Z(T)\delta W(t) &= Z(T)\int_0^T a(t)\mathrm{e}^{-\alpha t}\,dW(t) - \int_0^T a(t)\mathrm{e}^{-\alpha t}D_t Z(T)\,dt\\ &= Z(T)^2 - \int_0^T a^2(t)\mathrm{e}^{-2\alpha t}\,dt\ .\end{aligned}$$

This proves the result.

□

Note that

$$\mathbb{E}\big[Z(T)^2\big] = \int_0^T a^2(t)\mathrm{e}^{-2\alpha t}\,dt\ ,$$

by the Itô isometry. Consider the specific choice $a(t) = 2\alpha\mathrm{e}^{2\alpha t}/(\mathrm{e}^{2\alpha T}-1)$: Then

$$\begin{aligned}Z(T) &= \frac{2\alpha}{\mathrm{e}^{2\alpha T}-1}\int_0^T \mathrm{e}^{\alpha t}\,dW(t)\\ &= \frac{2\alpha}{\sigma(\mathrm{e}^{\alpha T}-1)}\big(\mathrm{e}^{2\alpha T}\ln S(T) - \ln x - \gamma(\mathrm{e}^{\alpha T}-1)\big)\ ,\end{aligned}$$

where $\gamma = \alpha(\mu-\lambda) - \sigma^2/2$. Furthermore,

$$\int_0^T a^2(t)\mathrm{e}^{-2\alpha t}\,dt = \frac{2\alpha}{(\mathrm{e}^{2\alpha T}-1)}\ .$$

The FD estimator for the gamma is given by

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}\ ,$$

where the same considerations to h as for the delta apply. In sec. 7 numerical tests are presented for the Malliavin approach and the FD approach to compare convergence speeds.

It is possible to derive the delta and the gamma by the density approach since the probability density of $X(t)$ is known. We state the result for the delta only:

Proposition 2.5 (Delta by the density approach)**.** *The delta of $u(x)$ can be represented as*

$$u'(x) \quad = \quad \frac{1}{x}E\big[f(S_T)\frac{2\alpha}{\sigma^2(e^{2\alpha T}-1)}\big(e^{\alpha T}\ln S(T) \;-\; \ln x \;-\; \gamma(e^{\alpha T} \;-\; 1)\big)\big] \quad . \tag{8}$$

Proof. Since

$$X(T) = \mathrm{e}^{-\alpha T}\ln x - \gamma(1-\mathrm{e}^{-\alpha T}) + \sigma\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW(s)\;,$$

we have that $X(T)$ is normally distributed with expectation $\mathrm{e}^{-\alpha T}\ln x - \gamma(1-\mathrm{e}^{-\alpha T})$ and variance $\sigma^2(1-\mathrm{e}^{-2\alpha T})/2\alpha$. Denoting the density by $\phi(z;x)$ (as a function of z), we find by straightforward differentiation with respect to x

$$\frac{\partial\phi}{\partial x}(z;x) = \phi(z;x)\frac{1}{x}\mathrm{e}^{-\alpha T}\frac{z-\gamma(1-\mathrm{e}^{-\alpha T})-\mathrm{e}^{-\alpha T}\ln x}{\frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha T})}\;.$$

Since differentiation and expectation commute in this case, we find

$$\begin{aligned}
u'(x) &= \frac{d}{dx}\int f(\mathrm{e}^z)\phi(z;x)\,dz\\
&= \int f(\mathrm{e}^z)\frac{\partial\phi}{\partial x(z;x)}\,dz\\
&= \int f(\mathrm{e}^z)\frac{1}{x}\mathrm{e}^{-\alpha T}\frac{z-\gamma(1-\mathrm{e}^{-\alpha T})-\mathrm{e}^{-\alpha T}\ln x}{\frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha T})}\phi(z;x)\,dz\;,
\end{aligned}$$

which yields the desired result. □

Note that the density approach leads to the same formula as in (7), which was derived using the Malliavin approach with a specific choice of the weight function $a(t)$.

We consider the vega for the European option on spot using the Malliavin approach:

Proposition 2.6 (Vega by the Malliavin approach)**.** *The vega of $u(\sigma)$ can be represented as*

$$u'(\sigma) \;=\; \sigma^{-1}E\big[f(S_T)\Big\{Z(T)\int_0^T e^{-\alpha(T-t)}\,dW_t - Z(T)\sigma(1-e^{-\alpha T}) - 1\Big\}\big]\;, \tag{9}$$

where $Z(T) = \int_0^T a(t)e^{\alpha(T-t)}\,dW_t$ *and* $a(t) \in \Gamma_T$.

Proof. The Malliavin derivative of S_T is given by

$$D_t S_T = S_T \sigma \mathrm{e}^{-\alpha(T-t)} \mathbf{1}_{t<T} \ .$$

By multiplying with a weight function $a(t) \in \Gamma_T$, and integrating each side, we find (after rearranging)

$$S_T = \sigma^{-1} \int_0^T a(t)\mathrm{e}^{\alpha(T-t)} D_t S_T \, dt \ .$$

The expression for the vega is now found by:

$$\begin{aligned}
u'(\sigma) &= \mathrm{E}\big[f'(S_T)\frac{dS_T}{d\sigma}\big] \\
&= \mathrm{E}\big[f'(S_T)S_T\big\{\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s - \sigma(1-\mathrm{e}^{-\alpha T})\big\}\big] \\
&= \mathrm{E}\big[\int_0^T f'(S_T)D_tS_T\sigma^{-1}\mathrm{e}^{\alpha(T-t)}a(t)\big\{\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s - \sigma(1-\mathrm{e}^{-\alpha T})\big\}\,dt\big] \\
&= \sigma^{-1}\mathrm{E}\big[\int_0^T D_tf(S_T)\mathrm{e}^{\alpha(T-t)}a(t)\big\{\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s - \sigma(1-\mathrm{e}^{-\alpha T})\big\}\,dt\big] \\
&= \sigma^{-1}\mathrm{E}\big[f(S_T)\int_0^T \mathrm{e}^{\alpha(T-t)}a(t)\big\{\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s - \sigma(1-\mathrm{e}^{-\alpha T})\big\}\,\delta W_t\big] \ .
\end{aligned}$$

The last stochastic integral δW_t is the Skorohod integral. Using the integration-by-parts formula for Skorohod integration, we get

$$\begin{aligned}
\int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s\,\delta W_t &= \int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t \cdot \int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s \\
&\quad - \int_0^T a(t)\mathrm{e}^{\alpha(T-t)}D_t\int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s\,dt \\
&= \int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t \cdot \int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s \\
&\quad - \int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\mathrm{e}^{-\alpha(T-t)}\mathbf{1}_{t<T}\,dt \\
&= \int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t \cdot \int_0^T \mathrm{e}^{-\alpha(T-s)}\,dW_s - 1 \ .
\end{aligned}$$

Hence,

$$u'(\sigma) = \sigma^{-1}\mathrm{E}\Big[f(S_T)\Big\{\int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t \cdot \int_0^T \mathrm{e}^{-\alpha(T-t)}\,dW_t \\ - \sigma(1-\mathrm{e}^{-\alpha T})\int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t - 1\Big\}\Big]\ .$$

$\square$

Choosing the weight function

$$a(t) = 2\alpha\mathrm{e}^{-2\alpha(T-t)}/(1-\mathrm{e}^{-2\alpha T})\ ,$$

yields, after some calculations,

$$\int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t \cdot \int_0^T \mathrm{e}^{-\alpha(T-t)}\,dW_t = \\ \frac{2\alpha}{\sigma^2(1-\mathrm{e}^{-2\alpha T})}\big(\ln S_T - \mathrm{e}^{-\alpha T}\ln S_0 - \gamma(1-\mathrm{e}^{-\alpha T})\big)^2$$

and

$$\sigma(1-\mathrm{e}^{-\alpha T})\int_0^T a(t)\mathrm{e}^{\alpha(T-t)}\,dW_t = \\ \frac{2\alpha(1-\mathrm{e}^{-\alpha T})}{(1-\mathrm{e}^{-2\alpha T})}\big(\ln S_T - \mathrm{e}^{-\alpha T}\ln S_0 - \gamma(1-\mathrm{e}^{-\alpha T})\big)\ .$$

Note that the chosen $a(t)$ gives

$$\begin{aligned} Z(T) &= \frac{2\alpha}{(1-\mathrm{e}^{-2\alpha T})}\int_0^T \mathrm{e}^{-\alpha(T-t)}\,dW_t \\ &= \frac{2\alpha}{(1-\mathrm{e}^{-2\alpha T})}\sigma^{-1}\big(\ln(S_T) - \mathrm{e}^{-\alpha T}\ln(S_0) - \gamma(1-\mathrm{e}^{-\alpha T})\big)\ , \end{aligned}$$

where we have used that $\int_0^T \mathrm{e}^{-\alpha(T-t)}\,dW_t = \sigma^{-1}(\ln(S_T) - \mathrm{e}^{-\alpha T}\ln(S_0) - \gamma(1-\mathrm{e}^{-\alpha T}))$. Repeated use of this in (9), and insertion of $Z(T)$ gives a computable expression for the vega.

The FD estimator for the vega is given analogous to the delta as

$$\frac{u(\sigma+h)-u(\sigma)}{h}\ .$$

3. Derivatives of European options on commodity and energy forwards

Consider a European option with maturity $\tau < T$ and payoff $f(F(\tau,T))$. The price of this option is

$$u = \mathbb{E}\big[f(F(\tau,T))\big] \,. \tag{10}$$

Like for derivatives of spot options, we shall use the notation $u(x(T))$ and $u(\sigma)$ to emphasize the parameters of interest. First, we are interested in calculating the delta of u, that is, the derivative with respect of $x(T)$. Strictly speaking, $x(T)$ is a function of T, the maturity of the forward contract, and the derivative should be interpreted as a functional derivative. However, we keep T fixed here, and therefore we may treat $du(X(T))/dx(T)$ as a standard derivative with respect to the variable $x(T)$. We denote this derivative $u'(x(T))$, which measures the sensitivity of u with respect to the initial forward price $x(T)$.

By the direct approach we find the expression (under the assumption that f is sufficiently regular so that differentiation can be moved inside the expectation, see e.g. [17], [12], [4] for conditions):

Proposition 3.1 (The direct approach)**.** *The delta of $u(x(T))$ can be represented as*

$$u'(x(T)) = \frac{1}{x(T)}\mathbb{E}\Big[f'(F(\tau,T))F(\tau,T)\Big] \,.$$

Proof. Direct differentiation gives (assuming sufficient regularity such that differentiation and expectation interchange),

$$u'(x(T)) = \mathbb{E}\Big[f'(F(\tau,T))\frac{d}{dx(T)}F(\tau,T)\Big] \,.$$

It is easily seen that $\frac{d}{dx(T)}F(\tau,T) = x^{-1}(T)F(\tau,T)$. □

The density of $F(\tau,T)$ is known, which means that we can differentiate with respect to the density function instead: By the density approach we find:

Proposition 3.2 (The density approach)**.** *The delta of $u(x(T))$ can be represented as*

$$u'(x(T)) = \frac{1}{x(T)}\mathbb{E}\Big[f(F(\tau,T))\Big(\frac{\ln(F(\tau,T)/x)}{\int_0^\tau \sigma^2(t,T)\,dt} + \frac{1}{2}\Big)\Big] \,.$$

Proof. We write $F(\tau,T)$ as

$$F(\tau,T) = \exp\Big(\ln x(T) - \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt + \int_0^\tau \sigma(t,T)\,dW(t)\Big) \,.$$

Since $\int_0^\tau \sigma(t,T)\,dW(t) \sim \mathcal{N}\big(0, \int_0^\tau \sigma^2(t,T)\,dt\big)$, we have

$$F(\tau,T) = \exp\Big(\ln x(T) - \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt + \epsilon \cdot \sqrt{\int_0^\tau \sigma^2(t,T)\,dt}\Big)\,,$$

where $\epsilon \sim \mathcal{N}(0,1)$ and the equality is in distribution. Hence,

$$u(x(T)) = \int_{\mathbb{R}} f(\mathrm{e}^z) g(z;x(T))\,dz$$

for the density function

$$g(z;x(T)) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(t,T)\,dt}} \exp\Big(-\frac{(z - \ln x(T) + \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt)^2}{2\int_0^\tau \sigma^2(t,T)\,dt}\Big)\,.$$

Differentiation of $g(z;x(T))$ with respect to $x(T)$ yields,

$$\frac{dg(z;x(T))}{dx(T)} = g(z;x(T)) \frac{z - \ln x(T) + \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt}{x(T)\int_0^\tau \sigma^2(t,T)\,dt}\,.$$

Hence,

$$\begin{aligned} u'(x(T)) &= \int_{\mathbb{R}} f(\mathrm{e}^z) g(z;x(T)) \frac{(z - \ln x(T) + \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt}{x\int_0^\tau \sigma^2(t,T)\,dt}\,dz \\ &= \mathbb{E}\Big[f(F(\tau,T))\frac{\ln(F(\tau,T)/x(T)) + \frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt}{x(T)\int_0^\tau \sigma^2(t,T)\,dt}\Big]\,. \end{aligned}$$

Thus, the proposition is proved. □

Finally, using the Malliavin approach we find

Proposition 3.3 (The Malliavin approach)**.** *The delta of $u(x(T))$ can be represented as*

$$u'(x(T)) = \frac{1}{x(T)}\mathbb{E}\Big[f(F(\tau,T))\int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)\Big]\,,$$

where $a \in \Gamma_\tau$.

Proof. We follow the argumentation in Fournié et al [9, Section 3.2]: Introduce the process $Y(t,T)$ by

$$Y(t,T) = \exp\Big(-\frac{1}{2}\int_0^\tau \sigma^2(t,T)\,dt + \int_0^\tau \sigma(t,T)\,dW(t)\Big)\,,$$

which yields the representation $F(t,T) = x(T)Y(t,T)$. Let us do some calculations with the Malliavin derivative of $F(t,T)$: Straightforward application of the Malliavin derivative yields

$$D_t F(\tau,T) = x(T)Y(\tau,T)\sigma(t,T)\mathbf{1}_{t<\tau}\,.$$

Rearranging,

$$Y(\tau,T)\mathbf{1}_{t<\tau} = x^{-1}(T)\sigma^{-1}(t,T)D_tF(\tau,T)\ .$$

Multiplying both sides with a function $a(t)\in\Gamma_\tau$, and then integrating from 0 to τ, gives,

$$Y(\tau,T) = x^{-1}(T)\int_0^\tau D_tF(\tau,T)a(t)\sigma^{-1}(t,T)\,dt\ .$$

Direct differentiation gives

$$\begin{aligned}
u'(x(T)) &= \mathbb{E}\big[f'(F(\tau,T))Y(\tau,T)\big]\\
&= x^{-1}(T)\mathbb{E}\big[\int_0^\tau f'(F(\tau,T))D_tF(\tau,T)a(t)\sigma^{-1}(t,T)\,dt\big]\\
&= x^{-1}(T)\mathbb{E}\big[\int_0^\tau D_tf(F(\tau,T))a(t)\sigma^{-1}(t,T)\,dt\big]\\
&= x^{-1}(T)\mathbb{E}\big[f(F(\tau,T))\int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)\big]\ ,
\end{aligned}$$

where we used the chain rule for Malliavin derivative in the second last equality and the duality between Malliavin differentiation and Skorohod integration in the last. □

We consider different choices of the function $a(t)$: Choose $a(t) = K\sigma(t,T)$, where $K = 1/\int_0^\tau \sigma(t,T)\,dt$. Then

$$u'(x(T)) = x^{-1}(T)\mathbb{E}\Big[f(F(\tau,T))\frac{W(\tau)}{\int_0^\tau \sigma(t,T)\,dt}\Big]\ . \tag{11}$$

A different choice could be $a(t) = K\sigma^2(t,T)$, where $K = 1/\int_0^\tau \sigma^2(t,T)\,dt$. Then

$$u'(x(T)) = \mathbb{E}\Big[f(F(\tau,T))\frac{\int_0^\tau \sigma(t,T)\,dW(t)}{\int_0^\tau \sigma^2(t,T)\,dt}\Big]\ , \tag{12}$$

which, after a slight rewriting, coincides with the delta obtained using the density method.

We are also interested in calculating the gamma of u, that is, the double derivative with respect of $x(T)$. Similar considerations as for the delta applies.

Proposition 3.4 (The Malliavin approach)**.** *The gamma of $u(x(T))$ can be represented as*

$$u''(x(T)) = x(T)^{-2}\mathbb{E}[f(F(\tau,T))\Big\{Z^2(\tau,T) - Z(\tau,T) - \int_0^\tau a^2(t)\sigma^{-2}(t,T)\,dt\Big\}]\ ,$$

where $Z(\tau,T) = \int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)$ and $a(t)\in\Gamma_\tau$.

Proof. Write the the delta as

$$u'(x(T)) = x(T)^{-1}\mathbb{E}[f(F(\tau,T))Z(\tau,T)] \ ,$$

where $Z(\tau,T) = \int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)$. Then the gamma is given by differentiation of delta with respect to the initial condition $x(T)$:

$$\begin{aligned}
u''(x(T)) &= -x(T)^{-2}\mathbb{E}[f(F(\tau,T))Z(\tau,T)] \\
&\quad + x(T)^{-1}\mathbb{E}[f'(F(\tau,T))\frac{F(\tau,T)}{x(T)}Z(\tau,T)] \\
&= -x(T)^{-2}\mathbb{E}[f(F(\tau,T))Z(\tau,T)] \\
&\quad + x(T)^{-2}\mathbb{E}[\int_0^\tau f'(F(\tau,T))D_tF(\tau,T)a(t)\sigma^{-1}(t,T)Z(\tau,T)\,dt] \\
&= -x(T)^{-2}\mathbb{E}[f(F(\tau,T))Z(\tau,T)] \\
&\quad + x(T)^{-2}\mathbb{E}[f(F(\tau,T))\int_0^\tau a(t)\sigma^{-1}(t,T)Z(\tau,T)\,\delta W(t)] \ ,
\end{aligned} \tag{13}$$

where we used the chain rule for Malliavin derivative in the second last equality and the duality between Malliavin differentiation and Skorohod integration in the last. Using that

$$Z(\tau,T) = \int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t), \quad D_tZ(\tau,T) = a(t)\sigma^{-1}(t,T)\mathbf{1}_{t<\tau} \ ,$$

the integration-by-parts formula of the Skorohod integral gives

$$\begin{aligned}
\int_0^\tau a(t)\sigma^{-1}(t,T)Z(\tau,T)\,\delta W(t) &= Z(\tau,T)\int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t) \\
&\quad - \int_0^\tau a(t)\sigma^{-1}(t,T)D_tZ(\tau,T)\,dt \\
&= Z(\tau,T)^2 - \int_0^\tau a^2(t)\sigma^{-2}(t,T)\,dt \ .
\end{aligned}$$

The final formula for the gamma therefore reads

$$\begin{aligned}
u''(x(T)) = x(T)^{-2}\mathbb{E}[f(F(\tau,T))\Big(\big(\int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)\big)^2 \\
- \int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t) - \int_0^\tau a^2(t)\sigma^{-2}(t,T)\,dt\Big)] \ .
\end{aligned}$$

□

In order to get an implementable expression for gamma, choose the weight function $a(t) = \sigma(t,T)/\int_0^T \sigma(t,T)\,dt$. Then

$$u''(x) = x(t)^{-2}\mathbb{E}[f(F(\tau,T))\big\{W_\tau^2C^2 - W_\tau C - \tau C^2\big\}] \ ,$$

where $C = \alpha e^{-\alpha T}(e^{\alpha\tau} - 1)/\sigma$.

4. Derivatives of Asian options on commodity and energy forwards

Define an Asian claim with maturity $\tau < T$,

$$u = \mathbb{E}\big[f\big(\int_0^\tau F(t,T)\,dt\big)\big]\ . \tag{14}$$

Assume $f \in L^2(\mathbb{R})$ and $\mathbb{E}\big[f\big(\int_0^\tau F(t,T)\,dt\big)^2\big] < \infty$. Here one of the drawbacks of the density approach becomes evident. The density of $\int_0^T F(t,T)\,dt$ is *not* explicitly known to us, so that the density approach is not applicable. Consider the Malliavin approach.

Proposition 4.1 (The Malliavin approach)**.** *The delta of* $u(x(T))$ *can be represented as*

$$u'(x(T)) = \mathbb{E}\Big[f\big(\int_0^\tau F(t,T)\,dt\big)X(\tau,T)\Big]\ ,$$

where

$$\begin{aligned}X(\tau,T) = \frac{2}{x(T)\int_0^\tau F(t,T)\,dt}\Big\{&\sigma^{-2}(\tau,T)F(\tau,T) - \sigma^{-2}(0,T)x(T)\\ &+ 2\int_0^\tau \sigma_t(t,T)\sigma^{-3}(t,T)F(t,T)\,dt\\ &+ \frac{\int_0^\tau \sigma^{-1}(t,T)F(t,T)\int_t^\tau \sigma(u,T)F(u,T)\,du\,dt}{\int_0^\tau F(t,T)\,dt}\Big\}\ .\end{aligned}$$

Proof. Direct differentiation, and integration-by-parts yield

$$\begin{aligned}u'(x(T)) &= \mathbb{E}\big[f'(\int_0^\tau F(t,T)\,dt)\int_0^\tau Y(t,T)\,dt\big]\\ &= \mathbb{E}\big[f'(\int_0^\tau F(t,T)\,dt)2\int_0^\tau Y(t,T)\int_t^\tau Y(s,T)\,ds\,dt\int_0^\tau Y(t,T)\,dt)^{-1}\big]\\ &= \mathbb{E}\big[\int_0^\tau f'(\int_0^\tau F(t,T)\,dt)2Y(t,T)\int_t^\tau Y(s,T)\,ds\,dt\int_0^\tau Y(t,T)\,dt)^{-1}\,dt\big]\ ,\end{aligned}$$

where $F(t,T) = x(T)Y(t,T)$. A straightforward calculation reveals

$$D_tF(s,T) = F(s,T)\sigma(t,T)\mathbf{1}_{\{t<s\}}\ .$$

Thus

$$\int_0^\tau D_tF(s,T)\,ds = x(T)\sigma(t,T)\int_t^\tau Y(s,T)\,ds\ .$$

We therefore have (using the properties of Malliavin derivative)

$$
\begin{aligned}
u'(x(T)) &= \mathbb{E}\big[\int_0^\tau f'(\int_0^\tau F(s,T)\,ds)\int_0^\tau D_tF(s,T)\,ds x^{-1}(T)\sigma^{-1}(t,T)2Y(t,T) \\
&\quad \times (\int_0^\tau Y(s,T)\,ds)^{-1}\,dt\big] \\
&= \mathbb{E}\big[\int_0^\tau f'(\int_0^\tau F(s,T)\,ds) D_t\int_0^\tau F(s,T)\,ds x^{-1}(T)\sigma^{-1}(t,T)2Y(t,T) \\
&\quad \times (\int_0^\tau Y(s,T)\,ds)^{-1}\,dt\big] \\
&= \mathbb{E}\big[\int_0^\tau D_t f(\int_0^\tau F(s,T)\,ds) 2Y(t,T)x^{-1}(T)\sigma^{-1}(t,T) \\
&\quad \times (\int_0^\tau Y(s,T)\,ds)^{-1}\,dt\big] \\
&= \mathbb{E}\big[f(\int_0^\tau F(s,T)\,ds)X(\tau,T)\big]\ ,
\end{aligned}
$$

where

$$
X(\tau,T) = \frac{2}{x(T)}\int_0^\tau Y(t,T)\sigma^{-1}(t,T)(\int_0^\tau Y(s,T)\,ds)^{-1}\,\delta W(t)\ .
$$

Let us calculate $X(\tau,T)$: Integration-by-parts for Skorohod integrals:

$$
\begin{aligned}
&\int_0^\tau \sigma^{-1}(t,T)Y(t,T)(\int_0^\tau Y(s,T)\,ds)^{-1}\delta W(t) \\
&\qquad = \int_0^\tau \sigma^{-1}(t,T)Y(t,T)\,dW(t)(\int_0^\tau Y(s,T)\,ds)^{-1} \\
&\qquad\qquad - \int_0^\tau \sigma^{-1}(t,T)Y(t,T)D_t(\int_0^\tau Y(s,T)\,ds)^{-1}\,dt \\
&\quad = \int_0^\tau \sigma^{-1}(t,T)Y(t,T)\,dW(t)(\int_0^\tau Y(s,T)\,ds)^{-1} \\
&\qquad + \int_0^\tau \sigma^{-1}(t,T)Y(t,T)(\int_0^\tau Y(s,T)\,ds)^{-2}\int_t^\tau Y(u,T)\sigma(u,T)\,du\,dt\ .
\end{aligned}
$$

Consider $\sigma^{-2}(t,T)F(t,T)$ and assume $\partial\sigma(t,T)/\partial t := \sigma_t(t,T)$ exists. The Itô Formula yields,

$$
\begin{aligned}
d\big(\sigma^{-2}(t,T)F(t,T)\big) &= -2\sigma^{-3}(t,T)\sigma_t(t,T)F(t,T)\,dt + \sigma^{-2}(t,T)dF(t,T) \\
&= -2\sigma^{-3}(t,T)\sigma_t(t,T)F(t,T)\,dt + \sigma^{-1}(t,T)F(t,T)\,dW(t)\ .
\end{aligned}
$$

Integrating both sides from 0 to τ, and inserting into the expression for $X(\tau,T)$, gives the desired result. $\square$

Let us consider the concrete choice of $\sigma(t,T)$ given in (5). In this case it is straightforward to see that

$$X(\tau,T) = \frac{2}{x(T)}\Big\{\mathrm{e}^{2\alpha T}\frac{\mathrm{e}^{-2\alpha\tau}F(\tau,T)-x(T)}{\sigma^2\int_0^\tau F(t,T)\,dt} + 2\alpha\mathrm{e}^{2\alpha T}\frac{\int_0^\tau \mathrm{e}^{-2\alpha t}F(t,T)\,dt}{\sigma^2\int_0^\tau F(t,T)\,dt} + \frac{\int_0^\tau \mathrm{e}^{\alpha t}F(t,T)\int_t^\tau \mathrm{e}^{-\alpha u}F(u,T)\,du\,dt}{(\int_0^\tau F(t,T)\,dt)^2}\Big\}\,.$$

In practice one is interested in Asian options where the averaging is taken over discrete dates, i.e. the arithmetic average Asian option. The payoff function will be

$$u = \mathbb{E}\big[f\big(\sum_{k=0}^n F(t_k,T)\big)\big]\,, \tag{15}$$

where $0 = t_0 < t_1 < \ldots < t_n \le T$. The delta and gamma are given as follows:

Proposition 4.2 (The Malliavin approach)**.** *The delta and gamma of $u(x(T))$ can be represented as*

$$u'(x(T)) = \frac{1}{x(T)}\mathbb{E}\Big[f\big(\sum_{k=0}^n F(t_k,T)\big)Z(t_n,T)\Big]$$

$$u''(x(T)) = x^{-2}(T)\mathbb{E}\Big[f\big(\sum_{k=0}^n F(t_k,T)\big)\Big\{Z^2(t_n,T) - Z(t_n,T) - \int_0^{t_n} a^2(t)\sigma^{-2}(t,T)\,dt\Big\}\Big]\,,$$

where $Z(t_n,T) = \int_0^{t_n} a(t)\sigma^{-1}(t,T)\,dW(t)$ and $a(t)$ is such that $\int_0^{t_1} a(t)\,dt = 1$, and $\int_{t_k}^{t_{k+1}} a(t)\,dt = 0$ for $k = 1,\ldots,n-1$.

Proof. Direct differentiation leads to

$$u'(x(T)) = \mathbb{E}\big[\sum_k f'(\sum_k F(t_k,T))Y(t_k,T)\big]\,,$$

where $Y(t_k,T) = x^{-1}(T)F(t_k,T)$. Since

$$D_tF(t_k,T) = F(t_k,T)\sigma(t,T)\mathbf{1}_{\{t<t_k\}}\,,$$

we find

$$Y(t_k,T)\mathbf{1}_{\{t<t_k\}} = x^{-1}(T)\sigma^{-1}(t,T)D_tF(t_k,T)\,.$$

Introducing a function $a(t)\in\Gamma_{t_k}$ for all $k = 1,\ldots,n$, and integrating both sides after multiplication with this function gives

$$Y(t_k,T) = x^{-1}(T)\int_0^{t_n} a(t)\sigma^{-1}(t,T)D_tF(t_k,T)\,dt\,.$$

Hence,

$$\begin{aligned} u'(x(T)) &= x^{-1}(T)\mathbb{E}\big[\int_0^{t_n} \sum_k f'(\sum_k F(t_k,T))D_tF(t_k,T)a(t)\sigma^{-1}(t,T)\,dt\big] \\ &= x^{-1}(T)\mathbb{E}\big[\int_0^{t_n} D_t f(\sum_k F(t_k,T))a(t)\sigma^{-1}(t,T)\,dt\big] \\ &= x^{-1}(T)\mathbb{E}\big[f(\sum_k F(t_k,T))\int_0^{t_n} a(t)\sigma^{-1}(t,T)\,dW(t)\big] \ . \end{aligned}$$

□

Here is an example of a function $a(t)$ satisfying the property in Prop. 4.2:

$$a(t) = \begin{cases} t_1^{-1}, & t \in [0,t_1), \\ t - \frac{1}{2}(t_{k+1}+t_k), & t \in [t_k,t_{k+1}), k = 1,\dots,n-1 \end{cases} .$$

For this a, the Itô integral inside the expression for $u'(x(T))$ becomes:

$$\begin{aligned} \int_0^{t_n} a(t)\sigma^{-1}(t,T)\,dW(t) = &\sum_{k=0}^{n-1} c_k \int_{t_k}^{t_{k+1}} \sigma^{-1}(t,T)\,dW(t) \\ &+ \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} t\sigma^{-1}(t,T)\,dW(t) \ , \end{aligned}$$

where

$$c_k = \begin{cases} t_1^{-1}, & k = 0 \\ -\frac{1}{2}(t_{k+1}+t_k), & k > 0 \end{cases} .$$

Define $X_k := c_k \int_{t_k}^{t_{k+1}} \sigma^{-1}(t,T)\,dW(t)$ and $Y_k = \int_{t_k}^{t_{k+1}} t\sigma^{-1}(t,T)\,dW(t)$. It is easily seen that $\{X_k\}_k$ are independent random variables. Likewise for $\{Y_k\}_k$. In addition, X_i and Y_j are independent for all $i \neq j$. Both are distributed as

$$X_k \sim \mathcal{N}\big(0, c_k^2 \int_{t_k}^{t_{k+1}} \sigma^{-2}(t,T)\,dt\big),$$

and

$$Y_k \sim \mathcal{N}\big(0, \int_{t_k}^{t_{k+1}} t^2\sigma^{-2}(t,T)\,dt\big).$$

The covariance between X_k and Y_k is

$$\mathrm{Cov}\big(X_k,Y_k\big) = c_k \int_{t_k}^{t_{k+1}} t\sigma^{-2}(t,T)\,dt \ .$$

A natural example to consider is $\sigma(t,T)$ as given in (5).

In [10] they show that the representation of the derivative is of minimal variance if the weight can be written as a functional of the underlying price process. We demonstrate how this is here; Use Itôs Formula on $a(t)\sigma^{-2}(t,T)\ln F(t,T)$ to obtain

$$\begin{aligned}
&d\big(a(t)\sigma^{-2}(t,T)\ln F(t,T)\big) = \\
&\quad \big\{a'(t)\sigma^{-2}(t,T) - 2a(t)\sigma^{-3}(t,T)\sigma_t(t,T)\big\}\ln F(t,T)\,dt \\
&\quad + a(t)\sigma^{-2}(t,T)\frac{1}{F(t,T)}\,dF(t,T) + \frac{1}{2}a(t)\sigma^{-2}(t,T)\frac{-1}{F(t,T)^2}\big(dF(t,T)\big)^2 \\
&= \big\{a'(t)\sigma^{-2}(t,T) - 2a(t)\sigma^{-3}(t,T)\sigma_t(t,T) - \frac{1}{2}a(t)\big\}\ln F(t,T)\,dt \\
&\quad + a(t)\sigma^{-1}(t,T)\,dW(t)\ .
\end{aligned}$$

Hence, integrating both sides and rearranging, we get,

$$\begin{aligned}
\int_0^{t_n} a(t)\sigma^{-1}(t,T)\,dW(t) &= a(t_n)\sigma^{-2}(t_n,T)\ln F(t_n,T) - a(0)\sigma^{-2}(0,T)x(T) \\
&\quad - \int_0^{t_n}\big\{a'(t)\sigma^{-2}(t,T) - 2a(t)\sigma^{-3}(t,T)\sigma_t(t,T) - \frac{1}{2}a(t)\big\}\ln F(t,T)\,dt\ .
\end{aligned}$$

This shows that the weight is a functional of the underlying process $F(t,T)$.

We now look at the vega of an Asian option. Choose the volatility to be $\sigma(t,T) = \sigma\alpha(t,T)$, and consider (14) as a function of σ, that is, $u(\sigma)$. We are now interested in calculating the *vega* of $u(\sigma)$, $u'(\sigma)$, using the Malliavin approach. We concentrate the calculation to the discrete Asian case:

$$u(\sigma) = \mathbb{E}\big[f(\sum_{k=0}^{n} F(t_k,T))\big]\ ,$$

where $0 = t_0 < t_1 < \ldots < t_n \le T$.

Proposition 4.3 (The Malliavin approach)**.** *The vega of u can be represented as*

$$u'(\sigma) = \mathbb{E}\Big[f\big(\sum_{k=0}^{n} F(t_k,T)\big)X(\{t_k\},T)\Big]$$

where

$$\begin{aligned}
X(\{t_k\},T) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{k=1}^{n}\Big\{&\int_{t_{k-1}}^{t_k} a(t)\alpha^{-1}(t,T)\,dW(t)\cdot \\
&\Big(\ln F(t_k,T) - \ln F(t_{k-1},T) - \frac{1}{2}\sigma^2\int_{t_{k-1}}^{t_k}\alpha^2(t,T)\,dt\Big)\Big\}\ ,
\end{aligned}$$

and $a(t)$ is a function such that $\int_{t_{k-1}}^{t_k} a(t)\,dt = 1$ for $k = 1,\ldots,n$.

Proof. Direct differentiation gives

$$u'(\sigma) = \mathbb{E}\big[f'(\sum_{k=0}^{n} F(t_k,T)) \sum_{k=1}^{n} F(t_k,T) Z(t_k,T)\big] ,$$

where

$$Z(t_k,T) = \int_0^{t_k} \alpha(t,T)\, dW(t) - \sigma \int_0^{t_k} \alpha^2(t,T)\, dt .$$

The Malliavin derivative of $F(t_k,T)$ is given by

$$D_t F(t_k,T) = \sigma\alpha(t,T)F(t_k,T)\mathbf{1}_{\{t<t_k\}} .$$

Define as in Fournié et al [9, Section 3.3]:

$$\beta_a(t) = \sum_{k=1}^{n} a(t)\big(Z(t_k,T) - Z(t_{k-1},T)\big)\mathbf{1}_{\{t_{k-1}<t<t_k\}} ,$$

where $\int_{t_{k-1}}^{t_k} a(t)\, dt = 1, k = 1,2,\dots,n$. This yields (note that $Z(0,T) = 0$),

$$\begin{aligned}
\int_0^T \sigma^{-1}\alpha^{-1}(t,T)D_tF(t_k,T)\beta_a(t)\, dt &= F(t_k,T)\int_0^{t_k} \beta_a(t)\, dt \\
&= F(t_k,T)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} a(t)\, dt\big(Z(t_i,T) - Z(t_{i-1},T)\big) \\
&= F(t_k,T)Z(t_k,T) .
\end{aligned}$$

Hence,

$$\begin{aligned}
u'(\sigma) &= \sigma^{-1}\mathbb{E}\big[\int_0^T \sum_{k=1}^{n} f'(\sum_{k=0}^{n} F(t_k,T))D_tF(t_k,T)\alpha^{-1}(t,T)\beta_a(t)\, dt\big] \\
&= \sigma^{-1}\mathbb{E}\big[\int_0^T D_t f(\sum_{k=0}^{n} F(t_k,T))\alpha^{-1}(t,T)\beta_a(t)\, dt\big] \\
&= \mathbb{E}\big[f(\sum_{k=0}^{n} F(t_k,T))\sigma^{-1}\int_0^T \alpha^{-1}(t,T)\beta_a(t)\, \delta W(t)\big] .
\end{aligned}$$

Note that $\beta_a(t)$ is $\mathcal{F}_{t_k}$-measurable for $t \le t_k$, and thus anticipating. We calculate the Skorohod integral using integration-by-parts formula:

$$\begin{aligned}
X(\{t_k\},T) &:= \sigma^{-1}\int_0^T \alpha^{-1}(t,T)\beta_a(t)\,\delta W(t) \\
&= \sigma^{-1}\sum_{k=1}^n \int_0^T \mathbf{1}_{\{t_{k-1}<t<t_k\}}\alpha^{-1}(t,T)a(t)\big(Z(t_k,T)-Z(t_{k-1},T)\big)\,\delta W(t) \\
&= \sigma^{-1}\sum_{k=1}^n \int_{t_{k-1}}^{t_k} a(t)\alpha^{-1}(t,T)\big(Z(t_k,T)-Z(t_{k-1},T)\big)\,\delta W(t) \\
&= \sigma^{-1}\sum_{k=1}^n \big(Z(t_k,T)-Z(t_{k-1},T)\big)\int_{t_{k-1}}^{t_k} a(t)\alpha^{-1}(t,T)dW(t) \\
&\quad - \sigma^{-1}\sum_{k=1}^n \int_{t_{k-1}}^{t_k} a(t)\alpha^{-1}(t,T)D_t\big(Z(t_k,T)-Z(t_{k-1},T)\big)\,dt\ .
\end{aligned}$$

But, since

$$Z(t_k,T)-Z(t_{k-1},T) = \int_{t_{k-1}}^{t_k}\alpha(t,T)\,dW(t) - \sigma\int_{t_{k-1}}^{t_k}\alpha^2(t,T)\,dt\ ,$$

we find (for $t \in [t_{k-1},t_k]$),

$$D_t\big(Z(t_k,T)-Z(t_{k-1},T)\big) = \alpha(t,T)\ .$$

Hence, the last sum of integrals above becomes

$$\sigma^{-1}\sum_{k=1}^n a(t)\,dt = \sigma^{-1}\sum_{k=1}^n 1 = \frac{n}{\sigma}\ .$$

Recalling that

$$\ln F(t_k,T) - \ln F(t_{k-1},T) - \frac{1}{2}\sigma^2\int_{t_{k-1}}^{t_k}\alpha^2(t,T)\,dt + \sigma\int_{t_{k-1}}^{t_k}\alpha(t,T)\,dW(t)\ ,$$

we find

$$Z(t_k,T)-Z(t_{k-1},T) = \frac{1}{\sigma}\Big(\ln F(t_k,T) - \ln F(t_{k-1},T) - \frac{1}{2}\sigma^2\int_{t_{k-1}}^{t_k}\alpha^2(t,T)\,dt\Big)\ .$$

Hence,

$$\begin{aligned}
X(\{t_k\},T) = \frac{1}{\sigma^2}\int_{t_{k-1}}^{t_k} & a(t)\alpha^{-1}(t,T)\,dW(t) \\
& \Big(\ln F(t_k,T) - \ln F(t_{k-1},T) - \frac{1}{2}\sigma^2\int_{t_{k-1}}^{t_k}\alpha^2(t,T)\,dt\Big) - \frac{n}{\sigma}\ .
\end{aligned}$$

□

Now consider a specific choice for $a(t)$; let for $t \in [t_{k-1}, t_k)$

$$a(t) = \alpha^2(t,T) / \int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\ .$$

Then,

$$\begin{aligned}\int_{t_{k-1}}^{t_k} a(t)\alpha^{-1}(t,T)\,dt &= \frac{\int_{t_{k-1}}^{t_k} \alpha(t,T)\,dW(t)}{\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt} \\ &= \frac{1}{\sigma}\Big(\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big)^{-1} \\ &\quad \times \Big(\ln F(t_k,T) - \ln F(t_{k-1},T) + \frac{1}{2}\sigma^2 \int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big)\end{aligned}$$

by using

$$\ln F(t_k,T) - \ln F(t_{k-1},T) = -\frac{1}{2}\sigma^2 \int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt + \sigma \int_{t_{k-1}}^{t_k} \alpha(t,T)\,dW(t)\ .$$

Hence,

$$\begin{aligned}X(\{t_k\},T) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{n} &\Big(\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big)^{-1} \\ &\times \Big(\ln F(t_k,T) - \ln F(t_{k-1},T) - \frac{1}{2}\sigma^2 \int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big) \\ &\times \Big(\ln F(t_k,T) - \ln F(t_{k-1},T) + \frac{1}{2}\sigma^2 \int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big)\ .\end{aligned}$$

Therefore,

$$X(\{t_k\},T) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{n}\Big(\frac{\big(\ln F(t_k,T) - \ln F(t_{k-1},T)\big)^2}{\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt} - \frac{\sigma^4}{4}\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt\Big)\ .$$

If we furthermore let $\alpha(t,T) = \mathrm{e}^{-\alpha(T-t)}$ for a constant α, we get

$$\int_{t_{k-1}}^{t_k} \alpha^2(t,T)\,dt = \frac{1}{2\alpha}\big(\mathrm{e}^{-2\alpha(T-t_k)} - \mathrm{e}^{-2\alpha(T-t_{k-1})}\big)\ ,$$

which inserted into the expression for $X(\{t_k\},T)$ gives

$$\begin{aligned}X(\{t_k\},T) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{n}\Big(&\frac{2\alpha\big(\ln F(t_k,T) - \ln F(t_{k-1},T)\big)^2}{\mathrm{e}^{-2\alpha(T-t_k)} - \mathrm{e}^{-2\alpha(T-t_{k-1})}} \\ &- \frac{\sigma^4}{8\alpha}\big(\mathrm{e}^{-2\alpha(T-t_k)} - \mathrm{e}^{-2\alpha(T-t_{k-1})}\big)\Big)\ .\end{aligned}$$

Finally, note that by choosing $n = 1$ we get the vega for a European forward option: Let $t_1 = \tau$;

$$u'(\sigma) = \frac{d}{d\sigma}\mathbb{E}\big[f(F(\tau,T))\big] = \mathbb{E}\big[f(F(\tau,T))X(\tau,T)\big] \ ,$$

where

$$X(\tau,T) = -\frac{1}{\sigma} + \frac{1}{\sigma^2}\Big\{\int_0^\tau a(t)\alpha^{-1}(t,T)\,dW(t) \\ \times \Big(\ln F(\tau,T) - \ln x(T) - \frac{1}{2}\sigma^2\int_0^\tau \alpha^2(t,T)\,dt\Big)\Big\}$$

and $\int_0^\tau a(t) = 1$.

5. Localized Malliavin approach for call options

If a localized Malliavin technique is to be applied, we have to specify the payoff function f prior to the deduction. We calculate here expressions using the payoff of a call option, i.e. we choose

$$f(x) = \max\big(x-K,0\big) = \big(x-K\big)^+ \ ,$$

where K is the contracted strike price. As argued in Fournié et al, variance reduction is achieved by using a *localized* Malliavin technique. Instead of using the Malliavin approach to calculate expressions for derivatives globally, they suggest to use the approach only around the singularity of the payoff function. That is, localize the payoff function $f(x)$ around $x = K$, and use the Malliavin approach on this piece of f.

More specifically, introduce a "smoothened" Heaviside function

$$H_a(x) = \begin{cases} 0, & x < K-a \\ \frac{x-(K-a)}{2a}, & K-a \le x \le K+a \\ 1, & x > K+a \end{cases} \ , \tag{16}$$

for a constant a (which is not necessarily small!). Introduce

$$g_a(x) = \int_{-\infty}^x H_a(y)\,dy = \begin{cases} 0, & x < K-a \\ \frac{1}{4a}\big(x-(K-a)\big)^2, & x-K \le x \le K+a \\ 1, & x > K+a \end{cases} \ .$$

Note that $g_a'(x) = H_a(x)$. Finally, let

$$f_a(x) = f(x) - g_a(x) = \big(x-K\big)^+ - g_a(x) \ ,$$

and notice that $f_a(x) = 0$ whenever $x < K-a$ or $x > K+a$. We will understand $f_a(x)$ as the *localized* version of $f(x)$.

Proposition 5.1. *The delta and gamma of a call on the spot is given resp. by*

$$\begin{aligned}
u'(x) &= \frac{1}{\sigma x}\mathbb{E}\big[f_a(S(T))Z(T)\big] + \frac{1}{x\,e^{\alpha T}}\mathbb{E}\big[H_a(S(T))S(T)\big] \\
u''(x) &= \frac{1}{\sigma^2 x^2(T)}\mathbb{E}\big[f_a(S(T))\big\{Z^2(T) - \sigma Z(T) - \int_0^T a^2(t)e^{-2\alpha t}\,dt\big\}\big] \\
&\quad + \frac{1}{x^2(T)\,e^{\alpha T}}\mathbb{E}\big[H_a'(S(T))\tau^2(T)e^{-\alpha T} - H_a(S(T))S(T)(1-e^{-\alpha T})\big]\ ,
\end{aligned}$$

where $Z(T) = \int_0^T a(t)e^{-\alpha t}\,dW(t)$ *and* $a(t) \in \Gamma_T$.

Proof. Represent $(x-K)^+ = f_a(x) + g_a(x)$. Use Prop. 2.3 on the first term and direct differentiation on the second to obtain

$$\begin{aligned}
u'(x) &= \frac{d}{dx}\mathbb{E}\big[f_a(S(T))\big] + \frac{d}{dx}\mathbb{E}\big[g_a(S(T))\big] \\
&= \frac{1}{\sigma x}\mathbb{E}\big[f_a(S(T))Z(T)\big] \\
&\quad + \mathbb{E}\big[g_a'(S(T))x^{-1}(T)\mathrm{e}^{-\alpha T}S(T)\big]\ .
\end{aligned}$$

In the last equality we used that $\frac{d}{dx}S(T) = x^{-1}(T)\mathrm{e}^{-\alpha T}S(T)$, which proves the formula for the delta.

Using Prop. 2.4 on the first term and direct differentiation on the second yields,

$$\begin{aligned}
u''(x) &= \frac{d^2}{d^2x}\mathbb{E}\big[f_a(S(T))\big] + \frac{d^2}{d^2x}\mathbb{E}\big[g_a(S(T))\big] \\
&= \frac{1}{\sigma^2 x^2(T)}\mathbb{E}\big[f_a(S(T))\big\{Z^2(T) - \sigma Z(T) - \int_0^T a^2(t)\mathrm{e}^{-2\alpha t}\,dt\big\}\big] \\
&\quad + \frac{d}{dx}\mathbb{E}\big[H_a(S(T))x^{-1}(T)\mathrm{e}^{-\alpha T}S(T)\big]\ .
\end{aligned}$$

Differentiation of the last expectation yields the formula for the gamma. □

Proposition 5.2. *The delta and gamma of a call on the forward is given resp. by*

$$\begin{aligned}
u'(x(T)) &= \frac{1}{x(T)}\mathbb{E}\big[f_a(F(\tau,T))Z(\tau,T)\big] + \frac{1}{x(T)}\mathbb{E}\big[H_a(F(\tau,T))F(\tau,T)\big] \\
u''(x(T)) &= \frac{1}{x^2(T)}\mathbb{E}\big[f_a(F(\tau,T))\big\{Z^2(\tau,T) - Z(\tau,T) - \int_0^\tau a^2(t)\sigma^{-2}(t,T)\,dt\big\}\big] \\
&\quad + \frac{1}{x^2(T)}\mathbb{E}\big[H_a'(F(\tau,T))F^2(\tau,T)\big]\ ,
\end{aligned}$$

where $Z(\tau,T) = \int_0^\tau a(t)\sigma^{-1}(t,T)\,dW(t)$ *and* $a(t) \in \Gamma_\tau$.

Proof. Using the Malliavin expression for delta and direct differentiation we get

$$\begin{aligned} u'(x(T)) &= \frac{d}{dx(T)}\mathbb{E}\big[f_a(F(\tau,T))\big] + \frac{d}{dx(T)}\mathbb{E}\big[g_a(F(\tau,T))\big] \\ &= \frac{1}{x(T)}\mathbb{E}\big[f_a(F(\tau,T))Z(\tau,T)\big] + \mathbb{E}\big[g_a'(F(\tau,T))\frac{d}{dx(T)}F(\tau,T)\big] \ . \end{aligned}$$

But $\frac{d}{dx(T)}F(\tau,T) = x^{-1}(T)F(\tau,T)$, which yields the delta formula.

Using the expression for the gamma we have

$$\begin{aligned} u''(x(T)) &= \frac{d^2}{d^2x(T)}\mathbb{E}\big[f_a(F(\tau,T))\big] + \frac{d^2}{d^2x(T)}\mathbb{E}\big[g_a(F(\tau,T))\big] \\ &= x^{-2}(T)\mathbb{E}\big[f_a(F(\tau,T))\big\{Z^2(\tau,T) - Z(\tau,T) - \int_0^\tau a^2(t)\sigma^{-2}(t,T)\,dt\big\}\big] \\ &\quad + \frac{d}{dx(T)}\Big(\frac{1}{x(T)}\mathbb{E}\big[H_a(F(\tau,T))F(\tau,T)\big]\Big) \ . \end{aligned}$$

Differentiating in the last expression yields the desired result for the gamma. □

Proposition 5.3. *The delta and gamma of a call on the Asian forward is given by resp.*

$$\begin{aligned} u'(x(T)) &= \frac{1}{x(T)}\Big\{\mathbb{E}\big[f_a\big(\sum_{k=0}^{n}F(t_k,T)\big)Z(t_n,T)\big] \\ &\qquad + \mathbb{E}\big[H_a\big(\sum_{k=}^{n}F(t_k,T)\big)\big(\sum_{k=0}^{n}F(t_k,T)\big)\big]\Big\} \\ u''(x(T)) &= x^{-2}(T)\mathbb{E}\Big[f_a\big(\sum_{k=0}^{n}F(t_k,T)\big)\Big\{Z^2(t_n,T) - Z(t_n,T) \\ &\quad - \int_0^{t_n} a^2(t)\sigma^{-2}(t,T)\,dt\Big\}\Big] + x^{-2}(T)\mathbb{E}\Big[H_a'\big(\sum_{k=0}^{n}F(t_k,T)\big)\big(\sum_{k=0}^{n}F(t_k,T)\big)^2\Big] \ , \end{aligned}$$

where $Z(t_n,T) = \int_0^{t_n} a(t)\sigma^{-1}(t,T)\,dW(t)$ *and* $a(t)$ *is an integrable function on* $[0,t_n]$ *satisfying* $\int_0^{t_1} a(t)\,dt = 1$ *and* $\int_{t_k}^{t_{k+1}} a(t)\,dt = 0$*,* $k = 1,\dots,n-1$*.*

Proof. Using Prop. 4.2 and the fact that

$$\frac{d}{dx(T)}F(t_k,T) = \frac{1}{x(T)}F(t_k,T) \ ,$$

give the result. (see proofs for standard calls on forwards above). □

6. The adaptive method

The adaptive method is introduced to enable better utilization of the sampling points from the QMC method. The principle is very simple: Use more simulation points in the parts of the domain where the integrand fluctuates than in parts of the domain where it is zero or flat (linear). In the papers [7] and [8] the current adaptive method was presented for multidimensional integrals, but in this paper we shall only need it for one-dimensional integration.

We give a brief overview of the adaptive method limited to one-dimensional problems: It is easy to construct a QMC estimator for the integral g over a part of the integration domain $\mathcal{D}$, and we therefore can construct a method to evaluate the integral over all of $\mathcal{D}$ as a sum of such estimated values. Let $\mathcal{D} = \cup_i D_i,\ \cap_i D_i = \emptyset$, $i = 1, \ldots, P$. Then

$$\begin{aligned} A &= \sum_{i=1}^{P} \int_{D_i} g(x)\, dx \\ &\approx \sum_{i=1}^{P} \frac{|D_i|}{\Delta L_i} \sum_{l=L_{i-1}}^{L_i - 1} g(x_l)\ , \end{aligned}$$

where $\Delta L_i = L_i - L_{i-1}$ and x_l is scaled such that $x_l \in D_i$ when $l \in [L_{i-1}, L_i]$. $|D_i|$ is to be understood as the length of D_i for integration of one-dimensional integrands, and the volume for multi-dimensional integrands. The adaptive algorithm should decide on the number of sub-domains and their sizes, that is P and $|D_i|$, $\forall i$. Furthermore, the algorithm has to pick the best set of sub-domains, and how many simulation points ΔL_i to use in each of them. Alternative approaches use information from the integrand to develop approximations of the integral in sub-domains with a deterministic approach rather than with simulation. This is done in e.g. [3], [11], [6], [20].

We have chosen to use a binary tree to represent the decomposition of the domain. Each node in the tree corresponds to a distinct part of the domain, and when we expand the tree we divide the domain represented by a node in two parts of equal size. For one dimensional integrands, the criterion we use to decide on division is simply to find the parts of the domain that contributes more than a preset amount to the overall variance of the estimator. The divide and conquer algorithm is terminated when the estimated variability in all sub-domains are less than a preset limit. We estimate the contribution to the variance from each sub-domain by the expression

$$C_i = \frac{|D_i|}{2}\Big(\frac{g(p_1^i) + g(p_2^i)}{2} - g(p_0^i)\Big)\ ,$$

where $|D_i|$ is the length of the sub-domain.

If the adaptive algorithm performs perfectly in accordance with the assumptions, the contribution from each sub-domain to the overall variance should be equal. Therefore $\sigma_i |D_i| = c$, $\forall i$ ideally. But even if the adaption process aspire to use the simulation points as effectively as possible, we get some sub-domains in which the measured variability is close to the preset limit, and some where the variability is considerably lower than the limit. To circumvent this behavior we use less simulation points in the sub-domains where the variability is low. Theoretically, the fraction for the optimal allocation of points in each sub-domain can be shown to be

$$q_i^* = \frac{r_i \sigma_i}{\sum_{l=1}^{P} r_l \sigma_l}, \ i = 1, \ldots, P \ , \tag{17}$$

where r_i is the probability for a point to be contained in each bin represented by D_i. Therefore $r_i = |D_i|$ in our setting. This leads to the allocation of simulation points by the relation

$$\Delta L_{i+1} = L \frac{\sigma_i |D_i|}{\sum_{l=1}^{P} \sigma_l |D_l|} \ , \tag{18}$$

where L is the total budget of simulation points. This approach, however, assume knowledge of all σ_i, and the adaptive approach does not provide this knowledge at the stage in the process where the contributions to the value of the total integral are calculated. Instead, we have chosen to implement a simpler approach to finding the number of simulation points in each bin. We use the relation

$$\Delta L_{i+1} = L \frac{\sigma_i}{C} \ , \tag{19}$$

where C is the stopping criterion for the adaption process on the variance estimates. This approach avoid the overhead by traversing the tree to collect the σ_l values, and in our tests the approach works well compared to using the same number of simulations in each bin.

For more details on the adaptive method, further variance reduction techniques and numerical simulations on multidimensional problems, see [7] and [8].

7. Numerical examples and comparison

In the examples presented below, we set the risk-free rate to zero ($r = 0$), use $\sigma = 0.3$ for the constant σ in (5), and calculate the different measures for at-the-money call options with three months left to maturity ($T = 0.25$). The options on the forward contracts are calculated on forwards with six months left to expiration. For the parameters of the Schwartz mean-reverting model we use $\alpha = 0.5, \mu = 5, \lambda = 1.6$. We need to find the strike prices giving us at-the-money options: For contracts on the forward this strike is simply given by $K = x(T)$

since $t \to F(t,T)$ is a martingale in the risk-free setting (recall that $x(T)$ is todays forward curve). For options on spot the at-the-money strike is given by

$$K = \mathbb{E}\big[S(T)\big] = \mathbb{E}\big[\exp(X(T))\big] = \exp\left(\mathbb{E}\big[X(T)\big] + \frac{1}{2}\operatorname{Var}[X(T)]\right)$$
$$= \exp\left(\mathrm{e}^{-\alpha T}\ln S(0) + \gamma(1-\mathrm{e}^{-\alpha T}) + \frac{\sigma^2(1-\mathrm{e}^{-2\alpha T})}{4\alpha}\right).$$

In the simulations we use $x(T) = 100$ for options on forwards, and $S(0) = 100$ for options on spot prices.

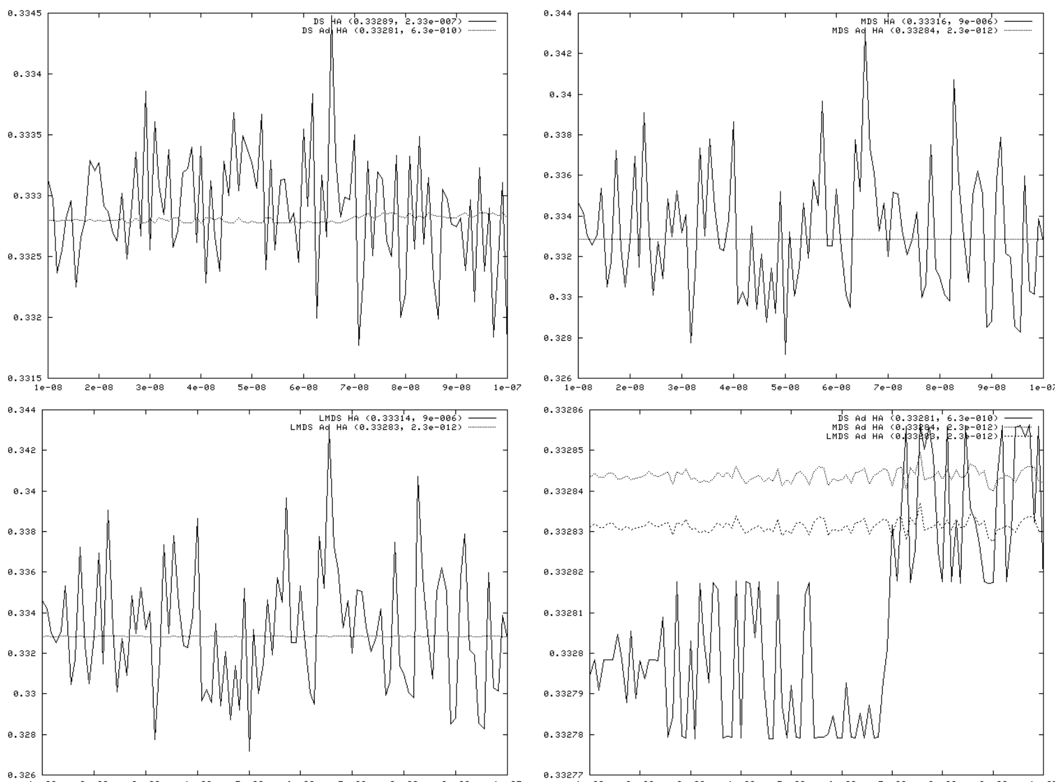

FIGURE 1. Delta for European option on commodity and energy spot

In the label of the plots the two numbers in the parenthesis are mean value and variance of the series of 100 estimators making up the plots. To label the plots we have used the abbreviations; **DS**: Delta Spot, **GS**: Gamma Spot, **VS**: Vega Spot, **DF**: Delta Forward, **GF**: Gamma Forward, **VF**: Vega Forward, **Ad**: Adaptive method is used, **M**: Malliavin approach is used, **LM**: Localized Malliavin approach is used, **HA**: The Halton Leaped method is used as low discrepancy sequence generator (used in all simulations in this article).

Note that when neither the Malliavin or the localized Malliavin method are used, the FD estimator is used. The given abbreviations are combined to indicate the

Table 1. Sensitivity parameters for European option on commodity and energy spot

Parameter	Method	Value (mean)	Variance
Delta	FD	0.33289	2.33e-007
Delta	FD Adaptive	0.33291	6.30e-010
Delta	Malliavin	0.33316	9.00e-006
Delta	Malliavin Adaptive	0.33284	2.30e-012
Delta	Local Malliavin	0.33314	9.00e-006
Delta	Local Malliavin Adaptive	0.33283	2.30e-012
Gamma	FD	0.01636	6.44e-003
Gamma	FD Adaptive	0.01205	1.29e-004
Gamma	Malliavin	0.015331	5.88e-008
Gamma	Malliavin Adaptive	0.015293	5.47e-014
Vega	FD	14.548	2.31e-003
Vega	FD Adaptive	14.539	5.67e-009
Vega	Malliavin	14.574	4.44e-001
Vega	Malliavin Adaptive	14.54	4.19e-008

Table 2. Sensitivity parameters for European option on commodity and energy forward

Parameter	Method	Value (mean)	Variance
Delta	FD	0.52483	7.16e-007
Delta	FD Adaptive	0.52479	1.19e-009
Delta	Malliavin	0.52491	1.86e-005
Delta	Malliavin Adaptive	0.52485	4.17e-012
Delta	Local Malliavin	0.52489	1.86e-005
Delta	Local Malliavin Adaptive	0.52482	1.30e-010
Gamma	FD	0.032778	2.07e-002
Gamma	FD Adaptive	0.027123	3.60e-004
Gamma	Malliavin	0.032031	1.96e-006
Gamma	Malliavin Adaptive	0.032022	1.87e-013
Vega	FD	16.527	2.87e-003
Vega	FD Adaptive	16.526	1.70e-008
Vega	Malliavin	16.531	5.23e-001
Vega	Malliavin Adaptive	16.527	4.98e-008

numerical experiment currently investigated. An example of an abbreviation is "LMDF_Ad HA", indicating that the simulation is performed by the localized Malliavin approach for the delta of a forward contract using the adaptive method and the Halton leaped sequence.

The number of samples for each of the 100 estimated values are in the range 800 to 1600, given as a result of the accuracy demanded of the adaptive method (this accuracy is given on the y-axis of the plots, and is in the range $[1E-7, 1E-8]$). For an MC-estimator this is a very low sample size, but for the adaptive method it is enough to reach the prescribed accuracy (see [7] for further details on the adaptive method). In [8] we find an estimator for comparing the efficiency of the adaptive approach with the conventional, where also the extended computing time of the adaptive method is taken into account. In one-dimensional problems, however, the extended computing time is small. Thus, the method with the lowest variance is preferred in the following cases.

In the last plot of the figs. 1 and 4, where the accurate estimators resulting from the adaptive method are compared, we see that some of the methods are biased. Based on the fact that the Malliavin approach gives an unbiased estimator, it is evident that the FD and the local Malliavin estimators are biased. This occurs also for the FD estimators of the calculations of vega in figs. 2 and 6. If we look at figs. 3 and 5 we see that the FD estimators are not able to capture the discontinuity in the first derivative and give an estimator with large variance.

8. Conclusion

The article deduces expressions for sensitivities of various derivative instruments in the commodity and energy market. The main focus is on the Malliavin approach, since we by this method can produce unbiased estimators under milder conditions on the payoff functions f than with conventional methods. The numerical results show that there are no apparent sacrifices connected to the Malliavin approach. The estimators of the Malliavin approach and the conventional FD approach have very similar convergence properties in the cases where they both exist, at least when the adaptive approach is employed to the problems.

The authors are currently working with numerical algorithms to use QMC methods on the estimators for the Asian contracts in the commodity and energy market presented in the paper. Some of the challenges lie in using the low discrepancy sequence in an optimal way. The results of these investigations will be reported elsewhere.

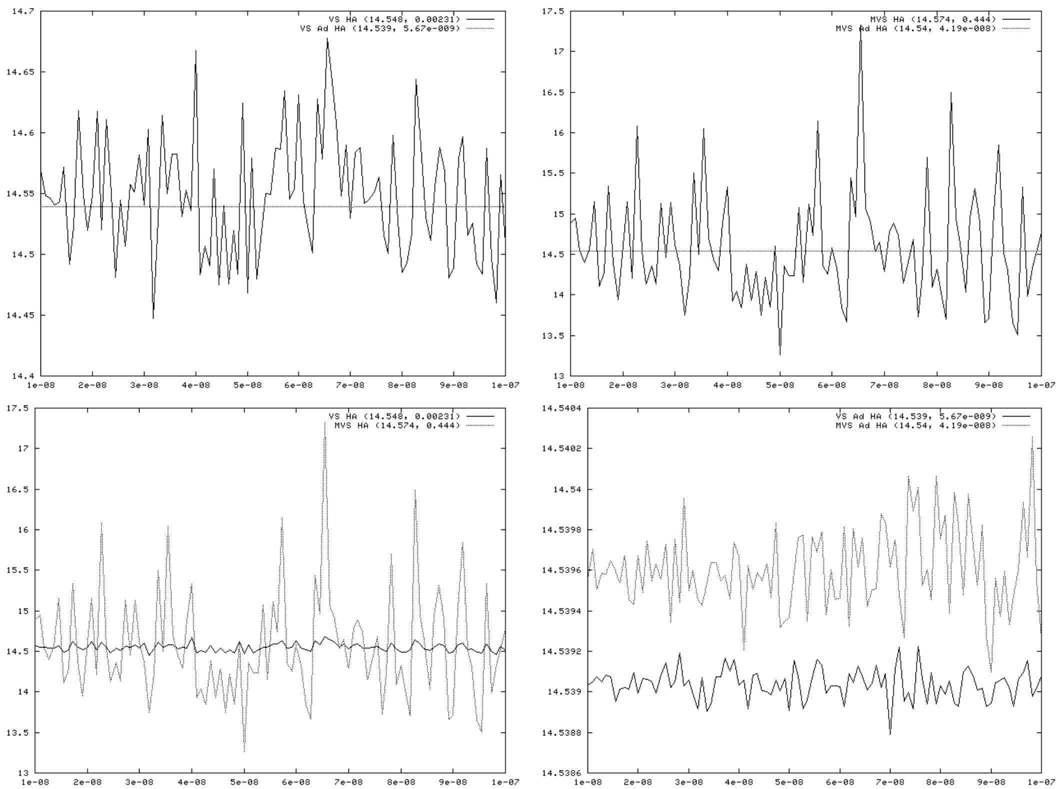

Figure 2. Vega for European option on commodity and energy spot

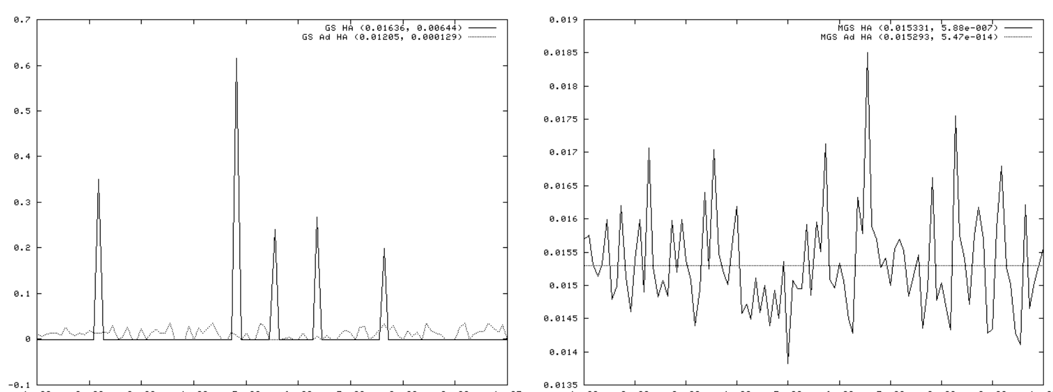

Figure 3. Gamma for European option on commodity and energy spot

References

[1] M. Avellaneda. An introduction to option pricing and the mathematical theory of risk. Technical report, Courant Institute-NYU and Morgan Stanley & Co., 1997.

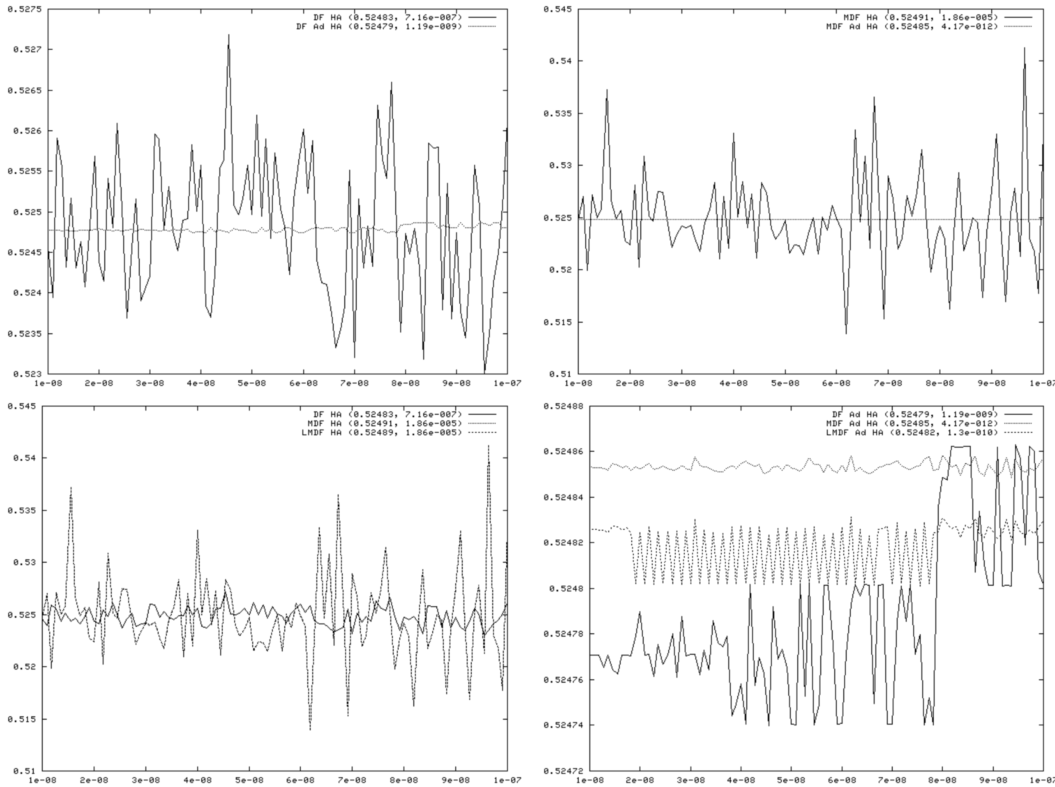

Figure 4. Delta for European option on commodity and energy forward

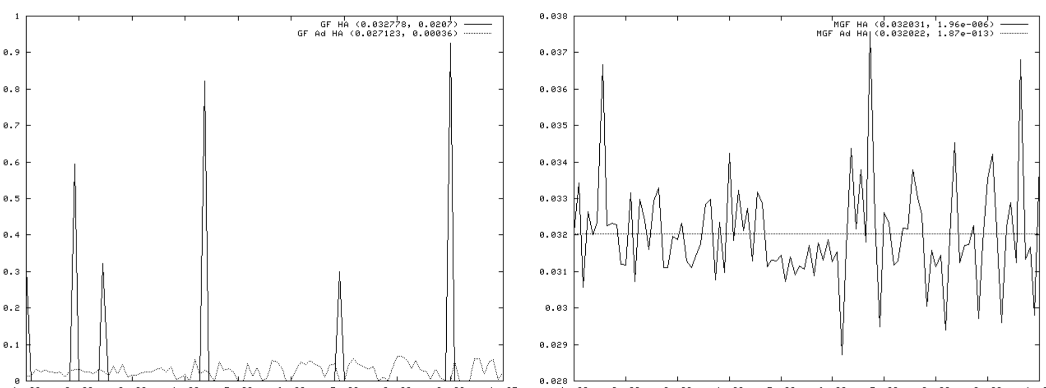

Figure 5. Gamma for European option on commodity and energy forward

[2] F. E. Benth. *Matematisk Finans - en Innføring i Opsjonsteori med Stokastisk Analyse*. Universitetsforlaget, 2002.

[3] J. Berntsen, T. O. Espelid, and A. Genz. An adaptive algorithm for the approximate calculation of multiple integrals. *ACM Transactions on Mathematical Software*, 1991.

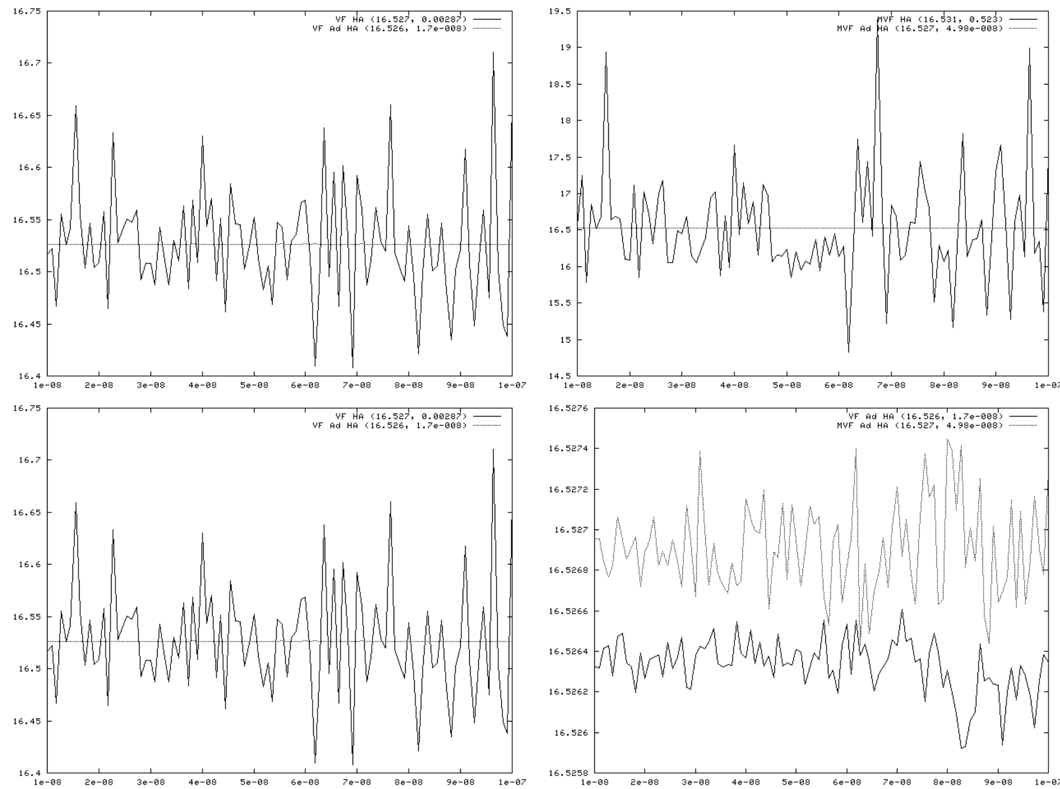

FIGURE 6. Vega for European option on commodity and energy forward

[4] M. Broadie and P. Glasserman. Estimating security price derivatives using simulation. *Management Science*, 42:269–285, 1996.

[5] L. Clewlow and C. Strickland. *Energy Derivatives. Pricing and Risk Management.* Lacima Publications, London, 2000.

[6] R. Cools. The approximation of low-dimensional integrals: available tools and trends. Technical report, Department of Computer Science, K.U.Leuven, May 1997.

[7] L. O. Dahl. An adaptive method for evaluating multidimensional contingent claims. Part I. Submitted to International Journal of Applied and Theoretical Finance. Currently in preprint at http://www.math.uio.no/eprint/pure_math/2002/pure_2002.html. , December 2001.

[8] L. O. Dahl. An adaptive method for evaluating multidimensional contingent claims. Part II. Accepted in the 8th International Vilnius Conference on Probability Theroy and Mathematical Statistics, June 2002. In preprint at http://www.math.uio.no/eprint/pure_math/2002/pure_2002.html. , February 2002.

[9] E. Fournié, J. M. Lasry, J. Lebuchoux, P. L. Lions, and N. Touzi. Application of malliavin calculus to Monte Carlo methods in finance. *Finance and Stochastics*, 3:391–412, 1999.

[10] E. Fournié, J. M. Lasry, J. Lebuchoux, P. L. Lions, and N. Touzi. Application of malliavin calculus to Monte Carlo methods in finance ii. *Finance and Stochastics*, 5:201–236, 2001.

[11] A. Genz and R. Cools. An adaptive numerical cubature algorithm for simplices. Technical report, Department of Computer Science, K.U.Leuven, December 1997.

[12] P. Glasserman. Gradient estimation via perturbation analysis. Kluwer Academic Press, Norwell, MA, 1991.
[13] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, 2:84–90, 1960.
[14] J. C. Hull. *Options Futures and Other Derivatives*. Prentice Hall, third edition edition, 1997.
[15] K. Kaaresen and E. Husby. A joint state-space model for electricity spot and futures prices. Report 965, Norwegian Computing Center, 2000.
[16] L. Kocis and W. J. Whiten. Computational investigations of low-discrepancy sequences. *ACM Transactions on Mathematical Software*, 23(2):266–294, June 1997.
[17] P. L'Ecuyer. An overview of derivative estimation. In *Proceedings of the 1991 Winter Simulation Conference*, pages 207–217, 1991.
[18] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer-Verlag, New York, 1995.
[19] D. Pilipović. *Energy Risk. Valuing and Managing Energy Derivatives*. McGraw-Hill, New York, 1998.
[20] W. H. Press and G. R. Farrar. Recursive stratified sampling for multidimensional Monte Carlo integration. *Computers in Physics*, 1990.
[21] E. S. Schwartz. The stochastic behaviour of commodity prices: Implications for valuation and hedging. *J. Finance*, LII(3):923–973, 1997.
[22] N. Taleb. *Dynamic Hedging*. John Wiley and Sons, New York, 1997.

(Fred Espen Benth)
Department of Mathematics
University of Oslo
PO Box 1053 Blindern, N–0316 Oslo, Norway
and
MaPhySto - Center for Mathematical Physics and Stochastics
University of Aarhus
Ny Munkegade
DK-8000 Århus, Denmark

E-mail address: fredb@math.uio.no

(Lars Oswald Dahl)
Department of Mathematical Sciences
Norwegian University of Science and Technology
N–7491 Trondheim, Norway
and
Storebrand Investments
PO Box 1380, N-0114 Oslo, Norway

E-mail address: lars.oswald.dahl@storebrand.com

(Kenneth Hvistendahl Karlsen)
Department of Mathematics
University of Bergen
Johs. Brunsgt. 12
N–5008 Bergen, Norway

E-mail address: kenneth.karlsen@mi.uib.no